

# An Entropy Functional for Riemann-Cartan Space-times

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**Abstract** By viewing space-time as a continuum elastic medium and introducing an entropy functional for its elastic deformations, T. Padmanabhan has shown that general relativity emerges from varying the functional and that the latter suggests holography for gravity and yields the Bekenstein-Hawking entropy formula. In this paper we extend this idea to Riemann-Cartan space-times by constructing an entropy functional for the elastic deformations of space-times with torsion. We show that varying this generalized entropy functional permits to recover the full set of field equations of the Cartan-Sciama-Kibble theory. Our generalized functional shows that the contributions to the on-shell entropy of a bulk region in Riemann-Cartan space-times come from the boundary as well as the bulk and hence does not suggest that holography would also apply for gravity with spin in space-times with torsion. It is nevertheless shown that for the specific cases of Dirac fields and spin fluids the system does become holographic. The entropy of a black hole with spin is evaluated and found to be in agreement with Bekenstein-Hawking formula.

**Keywords** Riemann-Cartan space-time · Entropy functional · Cartan-Sciama-Kibble field equations · Black holes · Holography.

## 1 Introduction

The study of black hole thermodynamics is certainly one of the most promising ways towards a deep understanding of the quantum nature of space and time. The pioneering works of Bekenstein [1,2] and Hawking [3,4], based respectively on the study of the properties of black holes geometry rising from general relativity and a semiclassical combination of the latter with quantum field theory, would later give birth to the novel concept of space-time holography [5,6]. It is then interesting to investigate the role of any possible extension of general relativity in the study of black hole thermodynamics in the hope of learning more about the nature of space-time at the quantum level.

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One of the old extensions of general relativity is the Einstein-Cartan theory of gravity. In this theory the intrinsic spin of a particle is naturally included in the geometric description of space-time using Riemann-Cartan geometry instead of the Riemannian geometry, i.e. by introducing torsion besides the metric as another degree of freedom for space-time. In this theory the Einstein field equations are replaced by what are commonly known as the Cartan-Sciama-Kibble (CSK) field equations, first discovered by Cartan [7], then independently rediscovered later by Kibble [8] and Sciama [9].

Ironically, the initial idea that led É. Cartan to generalize the Riemannian geometry and introduce the concept of torsion for space-time came from an analogy with the macroscopic concept of torsion in the physics of continuum media. This relation between space-time and continuum media stayed a mere analogy, though, up until 1967 when A. D. Sakharov proposed that general relativity may after all be just a low-energy approximation to the dynamics of space-time in the same sense that elasticity is an approximation to the microstructure of solids [10, 11, 12]. Recently, various authors have investigated the idea of an eventual elasticity of space-time either by applying concepts from elasticity theory to explore its dynamics [13, 14, 15] or by bringing novel interpretations to some fundamental concepts of modern cosmology such as inflation [16, 17, 18] and cosmic strings [19, 20, 21] by generalizing the three dimensional theory of defects to space-time.

Recently still, T. Padmanabhan has taken up this idea of elastic space-time from a thermodynamic viewpoint and introduced an entropy functional to be associated with the elastic deformations [22, 23, 24]. When extremized in accordance with the second law of thermodynamics, the entropy functional yields the equations of general relativity (see also [25]). Furthermore, the functional implies that the contributions to the on-shell entropy of a bulk region of space-time reside only on the boundary of the region in accordance with the holographic principle. In addition, the entropy of spinless black holes is found to be proportional to the area of their event horizon in exact agreement with Bekenstein-Hawking formula. The ability of this approach to reproduce both the fundamental equations of general relativity and main results from the thermodynamics of black holes suggests that extending the approach into Riemann-Cartan space-times may provide a simple way to include spin in the study of black hole thermodynamics. The question then is whether it would be possible to construct a generalized functional that reproduces the full set of the CSK field equations as well as familiar results from black hole thermodynamics. It is our aim in this paper to show that this is indeed possible.

The paper is organized as follows. In Sect. 2 we introduce an entropy functional for space-times with torsion and show how the CSK field equations emerge from varying the functional with respect to the deformation vector field. In Sect. 3 we use the CSK field equations to obtain a general form for the on-shell functional in which the boundary contributions are separated from the bulk ones. We then examine two specific cases of matter fields with spin embedded in space-time with torsion for which the functional takes exactly the form obtained by Padmanabhan for Riemannian space-times. Moving on to space-times with an event horizon, we show in Sect. 4 how the Bekenstein-Hawking entropy formula for a black hole with spin is recovered in our approach. We conclude this work with a discussion section to highlight and comment our main results.

## 2 The CSK Field Equations from an Entropy Functional

The motivation behind the introduction of an entropy functional by Padmanabhan [22, 23, 24] was to consider, in the spirit of Sakharov, space-time as a continuum medium subject at low-energy to elastic deformations characterized by the differential deformation vector field (or the displacement vector field)  $u^i(x) = \bar{x}^i - x^i$  ( $i = 0, \dots, 3$ ), where  $x^i$  and  $\bar{x}^i$  are coordinates in the elastic space-time before and after deformation respectively. Matter, through its energy-moment tensor  $T_{ij}$ , is viewed in this approach as a default inside the medium that spoils translational invariance in the field  $u^i$ . It contributes to the functional through a quadratic coupling with  $u^i$ . Further, in order not to get more than second-order differential equations when the functional is varied with respect to  $u^i$ , the former is constrained to contain at most first-order covariant derivatives of the latter. In [25] the same arguments were used to introduce a general expression for the functional but a different philosophy was adopted to arrive at its final form which we reproduce here to serve as the starting point for our construction of a generalized version:

$$S = \int d^4x \sqrt{-g} \left[ \alpha \left( (\nabla_i u_j)(\nabla^j u^i) - (\nabla_i u^i)^2 \right) + (\lambda g_{ij} + T_{ij}) u^i u^j \right]. \quad (1)$$

The constant  $\alpha$  was determined to be  $1/8\pi G$  and the scalar  $\lambda$  was constrained, using one of the Bianchi identities in Riemannian geometry, to be  $\frac{1}{2}\alpha R - \Lambda$  with  $R$  being the Ricci scalar and  $\Lambda$  a cosmological constant.

In order to generalize the above functional for Riemann-Cartan space-times in the presence of matter with non-zero spin we need to take into account torsion, characterized by the tensor  $Q_{ij}{}^k = \Gamma_{[ij]}{}^k = \frac{1}{2}(\Gamma_{ij}{}^k - \Gamma_{ji}{}^k)$ , as an additional degree of freedom in the geometry of space-time ( $\Gamma_{ij}{}^k$  being the affine connection.) On the other hand, to handle matter with spin we know that, besides the symmetric energy-momentum tensor  $T_{ij}$ , one must also take into account the contributions of the spin-angular momentum tensor  $\Sigma_{ijk}$  which is antisymmetric in its first two indices. Now, in our approach metric and torsion will be taken to be emergent properties of space-time rather than being fundamental dynamical quantities in the same sense proposed in [22, 23, 24] regarding the metric of Riemannian space-times. Therefore, instead of the usual general relativistic definition of the spin-angular momentum tensor  $\Sigma_{ijk}$  based on the variation with respect to torsion [26, 27, 28], it is the field-theoretic definition [29]

$$\Sigma_{ij}{}^k = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \partial_k \varphi^a} (M_{ij})^a{}_b \varphi^b \quad (2)$$

that should be adopted in our approach.  $(M_{ij})^a{}_b$  being  $d_r \times d_r$  matrices in some representation  $r$  of the Lorentz group generators acting on the fields  $\varphi^a$ <sup>1</sup>. From the tensor  $\Sigma_{ijk}$  one can build another tensor, also antisymmetric in its first two indices, called the spin-energy potential and related to  $\Sigma_{ijk}$  by [29]

$$\Psi_{ijk} = \Sigma_{ijk} + \Sigma_{ikj} + \Sigma_{kji}. \quad (3)$$

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<sup>1</sup> Here we use a different positioning of indices on  $\Sigma_{ijk}$  from the one used in [29]. Also, we shall use through out this paper letters from the middle of the Latin alphabet to denote holonomic space-time indices and letters from the beginning of the alphabet to denote both anholonomic tangent-space indices and group representation indices.

Hence the spin property of matter gives rise to an additional tensor that should also appear as, among other things, a translational-invariance breaking term inside the functional, i.e. in a term analogous to  $T_{ij}u^iu^j$  in (1).

Now the way to construct a scalar functional quadratic in the field  $u^i$  and containing first-order derivatives of this latter, using the various contributions coming from the tensors mentioned above, is to judiciously contract the different indices each tensor brings to the functional with those of the field  $u^i$  and its derivatives. To help us find the right combination among the many possibilities, we start with the form displayed in (1) and successively add terms brought by the new ingredients in such a way that our generalized functional reduces to (1) in the absence of spin.

Beginning with the translational-invariance breaking terms, the analogue of the quadratic term  $T_{ij}u^iu^j$  in (1) that could be constructed from  $\Psi_{ijk}$ , given its antisymmetry in the first two indices, is  $\Psi_{ijk}u^ju^k$  with still one more index to be contracted away. By dimensional analysis, however, we know that  $\partial_i\Psi^i{}_{jk}$  has the dimension of an energy-momentum tensor [29] while the torsion tensor  $Q_{ij}{}^k$  has the same dimension as that of a derivative (length<sup>-1</sup>). So the natural analogue of  $T_{ij}u^iu^j$  would be  $Q^i\Psi_{ijk}u^ju^k$ , where  $Q_i = Q_{ij}{}^j$  is the trace of torsion. Now there is still the possibility of producing a quadratic term by contracting two components of the field  $u^i$  in an antisymmetric way with  $Q^i\Psi_{ijk}$ . Indeed, if we introduce the oriented 2-surface  $w^{jk}$  constructed from the parallelogram formed by two components  $u^j$  and  $u^k$ , then  $w^{jk}$  is quadratic in the field  $u^i$  and antisymmetric. Hence we shall add  $Q^i\Psi_{i[jk]}w^{jk}$  as a third translational-invariance breaking term in the functional.

Next, note that in the absence of spin we had the quadratic term  $\lambda g_{ij}u^iu^j$  containing the contributions of the scalar curvature and the cosmological constant through  $\lambda$ . Since the additional feature brought into the geometry by torsion is antisymmetry, we expect the analogue of  $\lambda g_{ij}u^iu^j$  to be a quadratic term that would rise as an antisymmetric coupling with the field  $u^i$  and whose origin would be due to torsion. Hence we use again the oriented 2-surface  $w^{ij}$  and introduce the term  $\theta_{ij}w^{ij}$ . The tensor  $\theta_{ij}$  is antisymmetric, coming from torsion, and whose explicit form should be determined using constraints from geometry as we did for  $\lambda g_{ij}$  in the absence of spin [25]. More specifically, we shall use again one of the Bianchi identities but in Riemann-Cartan space-time.

Staying with the contributions of geometry, it is clear that other ways of introducing torsion contracted with the field  $u^i$  without the use of  $\Psi_{ijk}$  still exist. To guide us in the search for the right combination we simply proceed by building on the 'kinetic' term in (1) consisting of the first two terms of the integrand. Indeed, the simplest and most straightforward way to introduce a coupling between the fields  $Q_{ij}{}^k$  and  $u^i$  is to perform, in a way reminiscent of gauge theories, the following substitution  $\nabla_i \rightarrow \tilde{\nabla}_i = \nabla_i - Q_i$  where  $Q_i$  is the trace of  $Q_{ij}{}^k$ . Actually, a satisfactory justification for this seemingly *ad hoc* substitution will shortly emerge below.

Finally, by dimensional analysis we also learn that contracting the tensor  $\Psi_{ijk}$  with one component  $u^i$  and one covariant derivative  $\nabla^ju^k$  produces a scalar of the same dimension as the term  $T_{ij}u^iu^j$ . Therefore we also add into the functional the following covariant scalar  $\Psi_{ijk}u^i\nabla^ju^k$ . In fact, when written in terms of  $\Sigma_{ijk}$  using (3), this term displays both a symmetric and antisymmetric coupling with regard to the two indices of  $\nabla^ju^k$  as well as those of  $u^i$  and  $u^k$ . Furthermore, if we perform on this term the above mentioned substitution  $\nabla_i \rightarrow \tilde{\nabla}_i$  we see the emergence of precisely the quadratic

term  $Q^i \Psi_{ijk} u^j u^k$  which we introduced above from different motivations, so that all the terms nicely hang together at the end.

Putting all these terms together into a generalized entropy functional, this latter reads

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[ \alpha \left( (\tilde{\nabla}_i u_j) (\tilde{\nabla}^j u^i) - (\tilde{\nabla}_i u^i)^2 \right) + (\lambda g_{ij} + T_{ij}) u^i u^j \right. \\ \left. + (\Sigma_{ijk} + \Sigma_{ikj} + \Sigma_{kji}) u^i \tilde{\nabla}^j u^k + (\theta_{ij} - Q^k \Sigma_{ijk}) w^{ij} \right], \quad (4)$$

where we used the relation (3) to trade everywhere  $\Psi_{ijk}$  for  $\Sigma_{ijk}$  in order to be able later to compare our final results with those given in reference [26]. The detailed form we get when substituting the explicit expression of  $\tilde{\nabla}$  is

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[ \alpha \left( (\nabla_i u_j) (\nabla^j u^i) - (\nabla_i u^i)^2 \right) + (\lambda g_{ij} + T_{ij} + 2Q^k \Sigma_{kij}) u^i u^j \right. \\ \left. + (\Sigma_{ijk} + \Sigma_{ikj} + \Sigma_{kji} + 2\alpha Q_i \delta_{jk} - 2\alpha Q_k \delta_{ij}) u^i \nabla^j u^k + (\theta_{ij} - Q^k \Sigma_{ijk}) w^{ij} \right]. \quad (5)$$

In what follows we shall see that this functional reproduces both groups of field equations of Cartan-Sciama-Kibble theory if we demand that it be extremized for any deformation vector field  $u^i$ . To perform correctly the variation of (5) with respect to the field  $u^i$ , we symbolically write the oriented 2-surface  $w^{ij}$  of the parallelogram as  $u^i \wedge u^j$  in order to take care of the oriented product. Thus we have  $\delta w^{ij} = \delta u^i \wedge u^j + u^i \wedge \delta u^j = u^i \wedge \delta u^j - u^j \wedge \delta u^i$ . When this is done in the variation of the functional, the condition  $\delta \mathcal{S} = 0$  for all  $u^i$  yields

$$2\alpha \nabla_{[i} \nabla_{j]} u^j + (2\alpha Q_i \delta_{jk} - 2\alpha Q_k \delta_{ij} + \Sigma_{ikj}) \nabla^j u^k \\ + [\lambda g_{ij} + T_{ij} - \theta_{ij} - 2\alpha \nabla_{[i} Q_{j]} + \frac{1}{2}(\nabla^k + 2Q^k)(\Sigma_{ijk} + \Sigma_{kij} + \Sigma_{kji})] u^j = 0. \quad (6)$$

Next, one may write the second-order derivatives of  $u^i$  in (6) as a combination of first-order and zero-order derivatives if one uses the following contracted Ricci identity in Riemann-Cartan geometry [26]:  $2\nabla_{[i} \nabla_{j]} u^j = -R_{ij} u^j - 2Q_{ij}{}^k \nabla_k u^j$ . Hence identity (6) becomes

$$[2\alpha Q_{kij} + 2\alpha Q_i \delta_{jk} - 2\alpha Q_k \delta_{ij} + \Sigma_{ikj}] \nabla^j u^k \\ - [\alpha R_{ij} - \lambda g_{ij} - T_{ij} + \theta_{ij} + 2\alpha \nabla_{[i} Q_{j]} - \frac{1}{2}(\nabla^k + 2Q^k)(\Sigma_{ijk} + \Sigma_{kij} + \Sigma_{kji})] u^j = 0, \quad (7)$$

which, in turn, may be satisfied for any  $u^j$  if and only if each of the factors vanishes separately, that is

$$2\alpha(Q_{ijk} + Q_j \delta_{ik} - Q_i \delta_{jk}) = \Sigma_{ijk} \quad (8)$$

and

$$\alpha R_{ij} - \lambda g_{ij} = T_{ij} - \theta_{ij} - 2\alpha \nabla_{[i} Q_{j]} + \frac{1}{2}(\nabla^k + 2Q^k)(\Sigma_{ijk} + \Sigma_{kij} + \Sigma_{kji}). \quad (9)$$

We recognize in (8) the first group of the CSK field equations [26]. Now the second group of field equations of the CSK theory has also emerged and it resides in (9). This

can readily be seen as follows. From (8) one easily extracts the following identities:  $\nabla^k \Sigma_{ijk} = 4\alpha \nabla_{[i} Q_{j]} + 2\alpha \nabla^k Q_{ijk}$  and  $Q^k \Sigma_{ijk} = 2\alpha Q^k Q_{ijk}$ . Using these after taking the antisymmetric part of (9) and then comparing the result with the following Bianchi identity in Riemann-Cartan geometry [26]:  $R_{[ij]} = \nabla^k Q_{ijk} + 2Q^k Q_{ijk} + 2\nabla_{[i} Q_{j]}$ , immediately reveals what the antisymmetric tensor  $\theta_{ij}$  is:

$$\theta_{ij} = -2\alpha \nabla_{[i} Q_{j]}. \quad (10)$$

It is a tensor built from torsion as we argued above that it would be. Finally, substituting (10) into (9), this latter reads

$$\alpha(R_{ij} - \frac{1}{2}g_{ij}R) + \Lambda g_{ij} = T_{ij} + \frac{1}{2}(\nabla^k + 2Q^k)(\Sigma_{ijk} + \Sigma_{kij} + \Sigma_{kji}). \quad (11)$$

This is the second group of the CSK field equations comprising a cosmological constant.

Having built an entropy functional that reproduces the full set of the CSK field equations when extremized, our aim is to apply it to the evaluation of the entropy of black holes with non-zero spin in space-times with torsion. Before we do that, however, we shall first derive in the next section a general form for the on-shell functional and examine two specific cases of matter fields with non-zero spin.

### 3 The On-shell Entropy Functional

The study of the entropy functional we are going to conduct in this section will be on-shell, i.e. the CSK field equations will be assumed to be satisfied in the bulk region of the space-time considered, so that entropy is kept extremized in accordance with the second law of thermodynamics. When the functional (5) is integrated by parts and both groups of the CSK field equations (8) and (11) are taken into account along with the constraint (10) on  $\theta_{ij}$ , the entropy functional becomes

$$\begin{aligned} \mathcal{S} = \frac{1}{8\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} & \left[ \nabla_i (u^j \nabla_j u^i - u^i \nabla_j u^j - 8\pi G \Sigma_{jk}^i u^j u^k) \right. \\ & \left. - (2\nabla_{[i} Q_{j]} + 8\pi G Q^k \Sigma_{ijk}) w^{ij} \right]. \end{aligned} \quad (12)$$

Using Stokes' theorem on the first parentheses and the Bianchi identity on the second, the functional also takes the following form

$$\begin{aligned} \mathcal{S} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} n_i & (u^j \nabla_j u^i - u^i \nabla_j u^j - 8\pi G \Sigma_{jk}^i u^j u^k) \\ & + \frac{1}{8\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} (\nabla^k Q_{ijk} - R_{[ij]}) w^{ij}, \end{aligned} \quad (13)$$

where  $h$  is the determinant of the three-dimensional metric corresponding to the hypersurface  $\partial\mathcal{M}$  bounding the integration region  $\mathcal{M}$  of space-time, and  $n_i$  a unit vector normal to that hypersurface. We therefore conclude that the contributions to the on-shell entropy of a bulk region  $\mathcal{M}$  of Riemann-Cartan space-times do not reside exclusively on the boundary  $\partial\mathcal{M}$  but do partially come from the bulk region  $\mathcal{M}$  as well. We shall nevertheless expose below two specific cases of matter fields for which we recover an entropy whose contributions solely come from the boundary  $\partial\mathcal{M}$  and whose explicit expression coincides with the one found by Padmanabhan for Riemannian space-times. In what follows we shall first treat the case of the spin-half Dirac field and then examine the case of spin fluids.

### 3.1 The On-shell Functional in the Presence of a Dirac Field

The Lagrangian of a free Dirac field  $\psi$  in a curved space-time with torsion is [26]

$$\mathcal{L} = \frac{i}{2}(\bar{\psi}\gamma^k\psi_{;k} - \bar{\psi}_{;k}\gamma^k\psi) + \frac{i}{4}K_{jkl}\bar{\psi}\gamma^{[j}\gamma^k\gamma^{l]}\psi - m\bar{\psi}\psi, \quad (14)$$

where  $\psi_{;k} = \partial_k\psi + \frac{1}{4}\omega_{kab}\gamma^{[a}\gamma^{b]}\psi$  with  $\omega_{kab}$  being the Riemannian part of the tangent-space connection and  $K_{jkl} = Q_{jkl} + Q_{ljk} - Q_{klj}$  is the contortion tensor.  $\gamma_i$  are space-time Dirac gamma matrices related to the constant tangent-space gamma matrices  $\gamma_a$  through the Vielbeins  $\gamma_i = e_i^a\gamma_a$ . Using the definition (2) with  $M_{ij} = \frac{1}{2}\gamma_{[i}\gamma_{j]}$  for Dirac spinors [29], one finds that the spin-angular momentum tensor is given by

$$\Sigma_{ijk} = \frac{i}{4}\bar{\psi}\gamma_{[i}\gamma_j\gamma_{k]}\psi, \quad (15)$$

and hence is a totally antisymmetric tensor. As a consequence, the third term in the first parentheses of (12) vanishes. But when using (8) to express torsion in terms of  $\Sigma_{ijk}$  it follows that  $Q_{ijk}$  is also totally antisymmetric and hence traceless. So the content of the second parentheses in (12) also vanishes. Finally, recalling that in Riemann-Cartan space-times one uses in the covariant derivative an affine connection  $\Gamma_{ij}^k$  that is related to the Christoffel connection  $\{\overset{k}{ij}\}$  by [26]

$$\Gamma_{ij}^k = \{\overset{k}{ij}\} + Q_{ij}^k + Q_{ij}^k - Q_j^k{}_i, \quad (16)$$

one easily sees from the total antisymmetry of torsion that  $u^j\nabla_j u^i - u^i\nabla_j u^j = u^j\hat{\nabla}_j u^i - u^i\hat{\nabla}_j u^j$  where  $\hat{\nabla}$  is the covariant derivative in Riemannian space-times. Hence, the form of the entropy functional for a Riemann-Cartan space-time in the presence of a Dirac field reduces to that of space-time without torsion [22]

$$\mathcal{S} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} n_i (u^j \hat{\nabla}_j u^i - u^i \hat{\nabla}_j u^j). \quad (17)$$

The effects of torsion, and therefore of spin, are nevertheless still present; being encoded inside the deformation vector field  $u^i$ .

### 3.2 The On-shell Functional in the Presence of a Spin Fluid

Proceeding to the case of a space-time filled with a spin fluid we shall use the following standard definition [30,31]: By a spin fluid we mean a perfect fluid of density  $\rho$ , four-velocity  $v^i$ , and whose spin-angular momentum tensor  $\Sigma_{ij}^k$  is given by

$$\Sigma_{ij}^k = \Xi_{ij} v^k. \quad (18)$$

$\Xi_{ij}$  is the spin-density assumed to satisfy the so-called Frenkel condition  $\Xi_{ij} v^j = 0$ . Then from (8) we have  $Q_i = -2\pi G \Sigma_{ij}^j$ , whence we immediately see that the previous Frenkel condition produces a traceless torsion  $Q_i = 0$ . Integral (12) therefore reduces to

$$\mathcal{S} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} n_i (u^j \nabla_j u^i - u^i \nabla_j u^j - 8\pi G \Xi^{ij} v^k u_j u_k). \quad (19)$$

Further, the explicit expression one obtains for  $Q_{ijk}$  from (8) and (18) is  $4\pi G \Xi_{ij} v_k$ . Then, on using (16) we find that  $u^j \nabla_j u^i - u^i \nabla_j u^j = u^j \hat{\nabla}_j u^i - u^i \hat{\nabla}_j u^j + 8\pi G \Xi^{ij} v^k u_j u_k$  and so integral (19) gives back (17) as the on-shell entropy functional for space-time containing spin fluids.

#### 4 Entropy of Black Holes with Intrinsic Spin

Having obtained the general form of the on-shell entropy functional we now proceed to its evaluation in space-time with an event horizon created by a black hole with non-zero spin, i.e. either a rotating black hole or a static black hole with a non-zero quantum spin. For that purpose, we shall adopt the idea and follow the strategy that were introduced in [22,23,24] for treating event horizons in an approach that views space-time as a continuum elastic medium. The idea is to view the forward translation in time  $t \rightarrow t + \varepsilon$  as a byproduct of the deformation  $u^i$ , which reads  $x^i \rightarrow x^i + u^i$  on a spacelike hypersurface whose unit normal being  $u^i$ , whereas the event horizon is viewed as a singular point in the deformation field  $u^i$  on which this latter satisfies  $n_i u^i = 0$ . The strategy then consists of evaluating the functional in the Rindler-like space near the horizon.

According to (8), torsion is algebraically related to the spin angular momentum so that torsion is vanishing outside the sources. Hence integral (12) greatly simplifies when the boundary enclosing the hole is taken to be the event horizon, for then (12) reduces to

$$\mathcal{S} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} n_i (u^j \nabla_j u^i) = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} n_i c^i, \quad (20)$$

where we have introduced the four-acceleration  $c^i = u^j \nabla_j u^i$  of the field  $u^i$  [22]. The next step is to go to the Rindler-like space near the horizon to evaluate (20). For that, we need a solution to the CSK field equations, which reduce to Einstein vacuum equations outside the sources. Usually, when dealing with vacuum Einstein equations one exploits the underlying spherical symmetry and adopts the Schwarzschild metric which is the unique static spherical-symmetric solution according to Birkhoff's theorem. Dealing with non-zero spin black holes, however, we don't have spherical symmetry anymore but expect an axial-symmetric solution whose axis would coincide with that of spin. So our choice of space-time will be the Kerr space-time<sup>2</sup>. Note that, concerning black holes with a quantum spin, our choice of space-time matches that of [32] where Kerr space-time is identified as a geometric model for the spinning electron. Recall that the Kerr metric around a black hole is specified by the mass  $M$  and the angular momentum per unit mass  $a = J/M$  of the hole. For the case of a quantum spin, it is  $s$  that should play the role of the classical angular momentum  $J$ , i.e.  $a = s/M$ . Now the Kerr metric in Boyer-Lindquist coordinates may be written as follows [33]

$$\begin{aligned} dl^2 &= -\frac{\varrho^2 \Delta}{\xi^2} dt^2 + \frac{\varrho^2}{\Delta} dr^2 + \varrho^2 d\theta^2 + \frac{\xi^2}{\varrho^2} (d\phi - \Omega dt)^2 \sin^2 \theta, \\ \varrho^2 &= r^2 + a^2 \cos^2 \theta, \\ \Delta &= r^2 - 2GM r + a^2, \\ \xi^2 &= (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \\ \Omega &= \frac{2aGM r}{\xi^2}. \end{aligned} \quad (21)$$

<sup>2</sup> This choice of space-time is valid for a neutron black hole for example. More generally though, it will be the Kerr-Newman space-time when electric and/or magnetic charges are present. The arguments we use and the results we obtain however are also valid for these general cases.



From this metric which is singular for  $r_{\pm} = GM \pm \sqrt{G^2 M^2 - a^2}$  one deduces the radius of the event horizon  $r_H$ , the area of the horizon  $A_H$ , and the surface gravity  $\kappa$

$$\begin{aligned} r_H &= r_+, \\ A_H &= 4\pi(r_H^2 + a^2), \\ \kappa &= \frac{r_+ - r_-}{4GM r_+}. \end{aligned} \quad (22)$$

Before we proceed to the extraction of the Rindler-like metric, we would like to make the following remark. It is well-known that according to (21) one gets a naked singularity whenever  $a > GM$ . So one might worry about such a singularity when dealing with quantum spins given that for a Kerr electron [32], as well as for a nucleus, one has  $a = s/m \gg GM$ . For the case of macroscopic black holes, however, we are dealing not with one nucleus but with collapsed nuclear matter whose final spin scarcely averages to a non-zero value  $\langle s \rangle$ , so that the ratio  $s/M$  stays far below the dangerous limit of producing a naked singularity.

Now, to extract the metric of the near-horizon region we follow [33] and expand (21) in terms of the variable  $N^2 = 4(r - r_+)/(r_+ - r_-)$ , keeping only the leading contribution. Denoting by the subscript  $H$  all quantities evaluated on the horizon the Kerr metric takes, up to the conformal transformation  $g_{ij} \rightarrow \varrho_H^2 g_{ij}$ , the following Rindler-like form [33]

$$dt^2 \approx -N^2 \kappa^2 dt^2 + dN^2 + d\theta^2 + \frac{\xi_H^2}{\varrho_H^4} \sin^2 \theta d\tilde{\phi}^2 = -\tilde{N}^2 dt^2 + \frac{d\tilde{N}^2}{\kappa^2} + d\theta^2 + \frac{\xi_H^2}{\varrho_H^4} \sin^2 \theta d\tilde{\phi}^2, \quad (23)$$

where  $\tilde{N} = N\kappa$  and  $\tilde{\phi} = \phi - \Omega_H t$ . Having this simple form of the metric we can now proceed, following [22], to the evaluation of (20). Remembering that the horizon sits at  $N = 0$ , integral (20) reduces to

$$S = \frac{1}{8\pi G} \int_0^{\frac{2\pi}{\kappa}} d\tau \int_0^\pi \int_0^{2\pi} \tilde{N}(n_i c^i) \xi_H \sin \theta d\theta d\tilde{\phi}, \quad (24)$$

where we have integrated along the  $\frac{2\pi}{\kappa}$ -periodic Euclideanized time  $\tau = it$  that removes the conical singularity [33]. Next, we recall that for the Levi-Civita covariant derivative  $\tilde{\nabla}$  one has on the horizon for which  $N \rightarrow 0$  the following limit  $\tilde{N}(n_i c^i) \rightarrow \kappa$  [22, 23, 24] (see also [34].) Thereby, the above integral evaluates to

$$S = \frac{\pi \xi_H}{G} = \frac{A_H}{4G}, \quad (25)$$

thus recovering the Bekenstein-Hawking formula for black holes with spin. In our approach, this last result is valid for black holes with an intrinsic quantum spin as well as for black holes with a classical spinning, i.e. the rotating black holes.

## 5 Discussion

By extending the construction of the entropy functional for space-times with torsion viewed as continuum elastic media, in order to study their thermodynamics, we have built a functional using arguments related to the physics of continuum media and their defects. We saw that the functional remarkably reproduces exactly both groups

of the CSK field equations when demanding that entropy be extremized in accordance with the second law of thermodynamics for all possible deformations of space-time. Furthermore, the new terms that appeared in the generalized functional when torsion and spin are taken into account actually add a nice intuitive interpretation for the rising of space-time torsion from spin. Indeed, we see from (5) that in addition to the quadratic coupling with the deformation vector field through the simple product of two components of the latter, the spin-angular momentum tensor has another quadratic coupling that is orientation-dependent. More precisely, it couples to the oriented 2-surface area formed by two components of the deformation field. Thus, whereas the CSK field equations (8) give a quantitative description of how space-time torsion rises from spin, this new form of coupling in the functional adds the following qualitative description. While energy-momentum deforms space-time in analogy to a bowling ball that bends a two-dimensional rubber sheet when placed on it, spin-angular momentum deforms space-time by changing also the orientation of each deformed infinitesimal area in analogy to an elastic rod that torques when the relative orientation of its edges gets twisted. The term  $\Psi_{ijk} u^i \nabla^j u^k$  enhances this analogy since it represents a contribution that evokes the "moment" of the deformation gradient.

Now, although every term of the functional (5) was motivated using physical arguments, our construction remains heuristic and the question of uniqueness of integral (5) has to be addressed here. In what follows we shall argue that integral (5) is actually the unique functional that when varied recovers exactly the CSK field equations (8) and (11). In fact, suppose there are other terms that might be added inside the integral to make it more general and still leave unchanged equations (8) and (11). These additional terms, if any, could only appear in the second and third parenthesis of (5) since the structure of the first parenthesis is dictated by the functional in the absence of spin while the generality of the fourth parenthesis is already assured by the tensor  $\theta_{ij}$ . Hence, the additional terms would be of the form  $A_{ij} u^i u^j$  and  $B_{ijk} u^i \nabla^j u^k$ , where  $A_{ij}(Q, \Sigma)$  would be a symmetric tensor built quadratically (for dimensional reasons) from torsion and the spin-angular momentum while  $B_{ijk}(Q, \Sigma)$  would be a third rank tensor linear in torsion and spin-angular momentum. When these terms are added to (5) and this latter is varied, one learns that in order for (8) and (11) to remain valid, one should impose on  $B_{ijk}$  and  $A_{ij}$  the following constraints:  $B_{ijk} = B_{kji}$  and  $\frac{1}{2} \nabla^j B_{ijk} = A_{ik}$ . Although the first of these constraints is an algebraic one and may easily be achieved in the framework of the CSK theory by a simple combination of  $Q$ 's and  $\Sigma$ 's, the second constraint would bring an additional dynamical equation to the theory, and hence could not be kept without departing from the CSK theory in which dynamics emerges solely from (11). Thus, we conclude that integral (5) is the unique extension to Riemann-Cartan space-times of the entropy functional capable in the framework of our approach to lead exactly to the CSK theory.

This extension to Riemann-Cartan space-times of the functional also brings a new insight on the holographic principle concerning space-times with torsion. Indeed, in Sect. 3 we saw that contrary to the case of zero-spin, the entropy of the bulk region of space-time does not rise solely from the boundary of the region but gets contributions from the bulk as well. Hence, the fact that this same entropy when extremized yields the Einstein-Cartan gravity theory without any prior assumption of the latter suggests that, due to torsion, the holographic principle does not necessarily apply for gravity with spin in Riemann-Cartan space-times in general. We nevertheless saw two specific cases for which the system is indeed holographic and displays the same general formula for entropy found by Padmanabhan for Riemannian space-times. Now, this

unconformity with the holographic principle is not due primarily to torsion created by the spin angular momentum but more specifically to the trace of torsion that enters the bulk integral in (12). Space-times with torsion are thus holographic whenever the spin-angular momentum tensor of the system is traceless. This observation helps us better understand the relevance to holography of Dirac fields and spin fluids. Dirac fields and spin fluids make space-times with torsion holographic because of the tracelessness of their spin-angular momentum tensors giving rise to a traceless torsion. For Dirac fields this comes from the anticommutation of the gamma matrices whereas for spin fluids it is thanks to the Frenkel condition  $\Xi_{ij}v^j = 0$ . Conversely, denoting by  $\Sigma_i$  the trace of the spin-angular momentum tensor  $\Sigma_{ijk}$  then, since  $Q_i = -2\pi G\Sigma_i$ , we arrive upon using (12) at the following statement. If one ventures forth and adopts holography as a first principle then one would conclude from this that in space-times with torsion are allowed only systems whose spin-angular momentum tensor is either traceless or constrained to obey the following equation,  $2\nabla_{[i}\Sigma_{j]} + 8\pi G\Sigma^k\Sigma_{ijk} = 0$ .

Finally, in Sect. 4, applying our functional to space-times with an event horizon we saw that the Bekenstein-Hawking entropy formula is what is recovered for black holes with a classical spinning and/or a quantum spin. In the latter case, the explicit dependence on spin one gets from (25) is

$$S = 2\pi(GM^2 + \sqrt{G^2M^4 - s^2}) \approx 4\pi GM^2 - \frac{\pi s^2}{GM^2} + \mathcal{O}(s^4). \quad (26)$$

From this formula we see that in order for a black hole's quantum spin to bring a notable difference with respect to the thermodynamics of Schwarzschild black holes, the average quadratic spin of the nucleons composing the black hole must be of order  $\langle s^2 \rangle \sim G^2M^4$ . That is, if we denote by  $n$  the number of neutrons and by  $n_a$  the number of the elementary spins  $\hbar$  aligned inside the neutron star before it collapses into a black hole of mass, say,  $3M_\odot$ , the required fraction of aligned spins is  $\frac{n_a}{n} \sim \frac{3mGM_\odot}{\hbar c} \sim 10^{19}$  ( $m$  being the neutron mass.) Thus, even if all the nucleon spins inside the black hole were aligned, which is very plausible in gravity with torsion given that spin-torsion interaction induces spin alignment in the presence of strong gravity inside collapsed matter [35], formula (26) shows that the quantum spin is far from being able to bring any significant contribution to the thermodynamics of macroscopic black holes. (The above required ratio  $n_a/n$  exceeds unity whenever the black hole's mass exceeds  $\sim 10^{12}\text{kg}$ .) Of greater importance, however, is the fact that formula (26) displays at first-order a departure from the entropy of a Schwarzschild black hole by a term proportional to  $\langle s^2 \rangle$ . This simple fact may actually be taken as a hint for the consistency of our whole enterprise. Indeed, it is well-known [26, 27, 28, 29] that the predictions of Einstein-Cartan gravity theory depart from those of general relativity because of quadratic terms coming from spin-spin contact interactions  $\langle \Sigma_{ijk}\Sigma^{ijk} \rangle$ . Therefore, formula (26) and the reasoning that led to it represent, to some extent, an indirect test that gives real credit to our inclusion of spin à la Cartan in an elastic space-time.

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